Purely Periodic beta-Expansions in the Pisot Non-unit Case

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VALERIE BERTHÉ AND ANNE SIEGEL

Abstract. It is well known that real numbers with a purely periodic decimal expansion are rationals having, when reduced, a denominator coprime with 10. The aim of this paper is to extend this result to beta-expansions with a Pisot base beta which is not necessarily a unit. We characterize real numbers having a purely periodic expansion in such a base. This characterization is given in terms of an explicit set, called a generalized Rauzy fractal, which is shown to be a graph-directed self-affine compact subset of non-zero measure which belongs to the direct product of Euclidean and p-adic spaces.

Keywords: expansion in a non-integral base, Pisot number, beta-shift, beta-numeration, purely periodic expansion, self-affine set.

Let β be a Pisot number and $T_\beta : x \mapsto \beta x \pmod 1$ be the associated β-transformation. The aim of this paper is to characterize the real numbers x in $\mathbb{Q}(\beta) \cap [0, 1)$ having a purely periodic β-expansion.

It is well known that if β is a Pisot number, then real numbers that have an ultimately periodic β-expansion are the elements of $\mathbb{Q}(\beta)$ [Ber77, Sch80]. Thus real numbers x that have a purely periodic beta-expansion belong to $\mathbb{Q}(\beta)$. We present a characterization that involves conjugates of the algebraic number x, and can be compared to Galois' theorem for classical continued fractions.

Theorem 1. *Let β be a Pisot number. A real number $x \in \mathbb{Q}(\beta) \cap [0, 1)$ has a purely periodic beta-expansion if and only if x and its conjugates belong to an explicit subset in the product of Euclidean and p-adic spaces (see Figure 2.2 below); this set (denoted by $\widetilde{\mathcal{R}_\beta}$ and called generalized Rauzy fractal) is a graph-directed self-affine compact subset in the sense of* [MW88] *of non-zero measure; the primes p that occur are prime divisors of the norm of β.*

The scheme of the proof is based on a geometric representation of the two-sided β-shift (X_β, S). Our results and proof are inspired by [IR05, IS01, San02] in which a similar characterization of purely periodic expansions when β is a Pisot unit is proved. Note that a characterization of periods of periodic β-expansions in the Pisot quadratic unit case is given in [QRY05] using the dynamical and tiling properties of Rauzy fractals.

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Construction of the set $\widetilde{\mathcal{R}_\beta}$ (introduced in Theorem 1) is inspired by the geometric representation as generalized Rauzy fractals (also called atomic surfaces) of substitutive symbolic dynamical systems developed in [Sie03] in the non-unimodular case. In fact, a substitution σ is a non-erasing morphism of the free monoïd $\mathcal{A}^*$ and a substitutive dynamical system is a symbolic dynamical system generated by an infinite sequence which is a fixed point of a substitution. Furthermore, if the β-expansion of 1 in base β is finite (β is said to be a simple Parry number) and if its length coincides with the degree of β, then the set $\widetilde{\mathcal{R}_\beta}$ involved in our characterization is exactly the generalized Rauzy fractal that is associated in [Sie03] with the underlying β-substitution (in the sense of [Thu89, Fab95]).

Rauzy fractals were first introduced in [Rau82] in the case of the Tribonacci substitution (see Example 2.2 below), and then in [Thu89], in the case of the β-numeration associated with the Tribonacci number. Rauzy's construction was partly developed to exhibit explicit factors of the substitutive dynamical system under the Pisot hypothesis, as rotations on compact abelian groups. Rauzy fractals can more generally be associated with Pisot substitutions (see [BK06, CS01a, CS01b, IR06, Mes98, Mes00, Sie03] and surveys [BS05, BBLT06, Fog02]), as well as with Pisot β-shifts under the name of central tiles in the β-numeration framework [Aki98, AS98, Aki99, Aki00, BBK06].

There are mainly two Rauzy fractals construction methods. A first approach inspired by the seminal paper [Rau82] is based on formal power series, and is developed in [Mes98, Mes00], or in [CS01a, CS01b]. A second approach via iterated function systems (IFS) and generalized substitutions has been developed on the basis of ideas from [IK91] in [AI01, SAI01, HZ98, SW02, IR06], with special focus on the self-similar properties of Rauzy fractals. Here we combine both approaches: we define the set $\widetilde{\mathcal{R}_\beta}$ by introducing a representation map of the two-sided shift (X_β, S) based on formal power series, and prove that this set has non-zero Haar measure by splitting it into pieces that are solutions of an IFS.

Our construction is very similar to the algebraic construction of Markov symbolic almost finite-to-one covers of hyperbolic toral automorphisms provided by the two-sided β-shift exhibited in [LS05, Sch00] (see also [KV98, Sid01, Sid02, Sid03, BK05]). This latter approach is based on previous works performed in the golden ratio case in [Ver92], and later in the quadratic Pisot case in [SV98]. More generally, it falls in the *arithmetic dynamics* framework [Sid03]; the point is to provide explicit arithmetic codings of hyperbolic automorphisms of the torus (or solenoids in the non-unit case), i.e., symbolic codings such that relevant geometric features have a clear symbolic translation. Two types of symbolic dynamical systems are often used to obtain such codings, namely substitutive dynamical systems and β-shifts.

The idea in [Sch00, Sid01, Sid02, SV98, Ver92] is to expand points of the torus in power series in a base given by a homoclinic point. One point of the present construction is to introduce codings which work both for β-numerations and substitutive dynamical systems: the rôle played by the power series in base a homoclinic point is played here by the power series of Section 1 in the β-numeration case, or more generally, by power series involving the right and left normalized Perron-Frobenius eigenvectors in the substitutive case (for more details, see [BS05, BBLT06]). This construction thus has consequences for the effective construction of Markov partitions for toral automorphisms whose main eigenvalue is a Pisot number. See, e.g. [Ber99, BK05, IO93, KV98, Pra99, Sie00].

The aim of this paper is thus twofold. We first want to characterize real numbers having a purely periodic β-expansion, and secondly, we try to perform the first steps of a study of the geometric representation of β-shifts in the Pisot non-unit case, generalizing the results of [Aki98, AS98, Aki99, Aki00], based on the formalism introduced in the substitutive case in [Sie03].

This paper is organized as follows. We first recall in Section 1 the basic elements required for β-expansions. We then associate in Section 2 with the two-sided β-shift (X_β, S) formal power series in $\mathbb{Q}[[X]]$; we obtain in Section 2.3 a representation map for the two-sided β-shift by gathering the set of finite values which can be taken for any topology (Archimedean or not) by these formal power series when specializing them in β: in fact, we take the completion of $\mathbb{Q}(\beta)$ with respect to all absolute values on $\mathbb{Q}(\beta)$ which take a value differing from 1 on β (this value is thus smaller than 1 since β is a Pisot number). We are then able to define the Rauzy geometric representation of the two-sided β-shift (Definition 3). Section 3 is devoted to a study of the properties of the set $\widetilde{\mathcal{R}_\beta}$. We then prove Theorem 1 in Section 4.

1. β-numeration

Let $\beta > 1$ be a real number. *In all that follows, β is assumed to be a Pisot number.* The Renyi β-expansion of a real number $x \in [0, 1)$ is defined as the sequence $(x_i)_{i\geq 1}$ with values in $\mathcal{A}_\beta := \{0, 1, \dots, [\beta]\}$ produced by the β-transformation $T_\beta : \ x \mapsto \beta x \pmod 1$ as follows:

$$\forall i \geq 1,\ u_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor, \text{ and thus } x = \sum_{i\geq 1} u_i \beta^{-i}.$$

Let $d_\beta(1) = (t_i)_{i\geq 1}$ stand for the β-expansion of 1. Numbers β such that $d_\beta(1)$ is ultimately periodic are called *Parry numbers* and those such that $d_\beta(1)$ is finite are called *simple Parry numbers*. Since β is assumed to be a Pisot number, then β is either a Parry number or a simple Parry number according to [Ber77]. Let $d^*_\beta(1) = d_\beta(1)$, if $d_\beta(1)$ is infinite, and $d^*_\beta(1) = (t_1 \dots t_{n-1}(t_n - 1))^\infty$, if $d_\beta(1) = t_1 \dots t_{n-1} t_n$ is finite ($t_n \neq 0$). The set of β-expansions of real numbers in $[0, 1)$ is exactly the set of sequences $(u_i)_{i\geq 1}$ in $\mathcal{A}_\beta^{\mathbb{N}}$, such that

$$\forall k \geq 1,\ (u_i)_{i\geq k} <_{\text{lex}} d^*_\beta(1). \tag{1.1}$$

For more details on the β-numeration, see for instance [Fro02, Fro00].

The (two-sided symbolic) β-shift. Let (X_β, S) be the two-sided symbolic dynamical system associated with β, where the shift map S maps the sequence $(y_i)_{i\in\mathbb{Z}}$ onto $(y_{i+1})_{i\in\mathbb{Z}}$. The set X_β is defined as the set of two-sided sequences $(y_i)_{i\in\mathbb{Z}}$ in $\mathcal{A}_\beta^{\mathbb{Z}}$ such that each left truncated sequence is less than or equal to $d^*_\beta(1)$, that is,

$$\forall k \in \mathbb{Z},\ (y_i)_{i\geq k} \leq_{\text{lex}} d^*_\beta(1). \tag{1.2}$$

We will use the following notation for the elements of X_β: if $y = (y_i)_{i\in\mathbb{Z}} \in X_\beta$, define $u = (u_i)_{i\geq 1} = (y_i)_{i\geq 1}$ and $w = (w_i)_{i\geq 0} = (y_{-i})_{i\geq 0}$. One thus gets a two-sided sequence of the form

$$\dots w_3 w_2 w_1 w_0 u_1 u_2 u_3 \dots$$

which is written as $y = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) = (w, u)$. In other words, we will use the letters (w_i) for the "past" and (u_i) for the "future" of the element $y = (w, u)$ of the two-sided shift X_β.

One-sided β-shifts. Let X^r_β denote the set of one-sided sequences $u = (u_i)_{i\geq 1}$ such that there exists $w = (w_i)_{i\geq 0}$ with $(w, u) \in X_\beta$. This set is called the *right one-sided β-shift*. It coincides with the usual one-sided β-shift and is equal to the set of sequences $(u_i)_{i\geq 1}$ which satisfy

$$\forall k \geq 1,\ (u_i)_{i\geq k} \leq_{\text{lex}} d^*_\beta(1). \tag{1.3}$$

Similarly, X^l_β is defined as the set of one-sided sequences $w = (w_i)_{i\geq 0}$ such that there exists $u = (u_i)_{i\geq 1}$ with $(w, u) \in X_\beta$. We call this set the *left one-sided β-shift*.

Sofic shift. Since β is a Parry number (simple or not), (X_β, S) is sofic [BM86]. A finite word $w_1 \cdots w_k$ is said to be a *factor* of the sequence $(v_i)_{i\in\mathbb{Z}}$ if there exists an index ℓ such that $w_1 = v_\ell, w_2 = v_{\ell+1}, \cdots, w_k = v_{\ell+k-1}$. We denote by $F(X_\beta)$ the set of finite factors of the sequences in X_β; the minimal automaton $\mathcal{M}_\beta$ recognizing the set of factors of $F(D_\beta)$ can easily be constructed (see Figure 1.1). The number of states d of this automaton is equal to the length of the period n of $d^*_\beta(1)$ if β is a simple Parry number with $d_\beta(1) = t_1 \dots t_{n-1} t_n$, $t_n \neq 0$, and to the sum of its preperiod n plus its period p, if β is a non-simple Parry number with $d_\beta(1) = t_1 \dots t_n (t_{n+1} \dots t_{n+p})^\infty$ ($t_n \neq t_{n+p}$, $t_{n+1} \cdots t_{n+p} \neq 0^p$).

β-substitutions. Let us recall that a *substitution* σ is a morphism of the free monoid $\mathcal{A}^*$, such that the image of each letter of $\mathcal{A}$ is non-empty. As introduced for instance in [Thu89] and [Fab95], one can naturally associate a substitution σ_β (called β-substitution) with (X_β, S) over the alphabet $\{1, \cdots, d\}$, where d stands for the number of states of the automaton $\mathcal{M}_\beta$: j is the k-th letter occurring in $\sigma_\beta(i)$ (that is, $\sigma_\beta(i) = pjs$, where $p, s \in \{1, \cdots, d\}^*$ and the length of p is equal to $k-1$) if and only if there is an

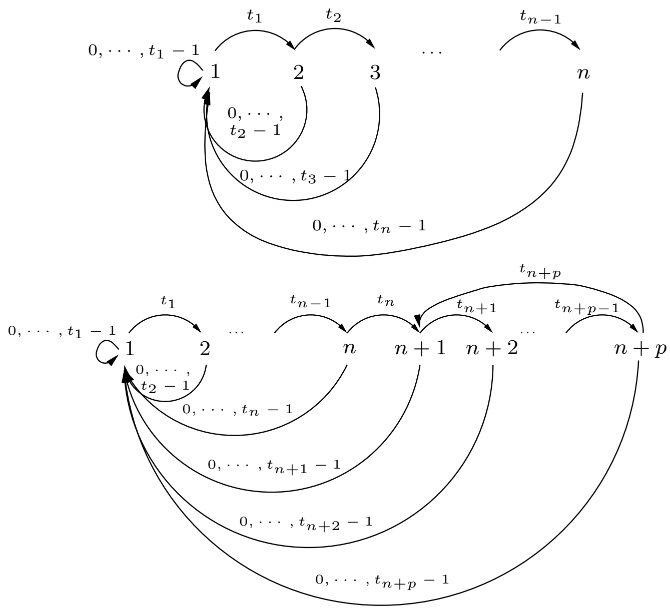

FIGURE 1.1. The automata $\mathcal{M}_\beta$ for a simple Parry number ($d_\beta(1) = t_1 \dots t_{n-1} t_n$) and for a non-simple Parry number ($d_\beta(1) = t_1 \dots t_n (t_{n+1} \dots t_{n+p})^\infty$).

arrow in $\mathcal{M}_\beta$ from the state i to the state j labeled by $k-1$. This definition is easily found to be consistent.

An explicit formula for σ_β can be computed by considering the two different cases, β simple and β non-simple Parry number.

- Assume $d_\beta(1) = t_1 \dots t_{n-1} t_n$ is finite, with $t_n \neq 0$. Thus $d^*_\beta(1) = (t_1 \dots t_{n-1}(t_n - 1))^\infty$. One defines σ_β over the alphabet $\{1, 2, \dots, n\}$ as shown in (1.4).
- Assume that $d_\beta(1)$ is infinite. Then it cannot be purely periodic (according to Remark 7.2.5 [Fro02]). Hence one has $d_\beta(1) = d^*_\beta(1) = t_1 \dots t_n (t_{n+1} \dots t_{n+p})^\infty$, with $n \geq 1$, $t_n \neq t_{n+p}$ and $t_{n+1} \dots t_{n+p} \neq 0^p$. One defines σ_β over the alphabet $\{1, 2, \dots, n+p\}$ as shown in (1.4).

(1.4)

$$\sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1}2 \\ 2 & \mapsto 1^{t_2}3 \\ \vdots & \vdots \\ n-1 & \mapsto 1^{t_{n-1}}n \\ n & \mapsto 1^{t_n}. \end{cases} \qquad \sigma_\beta : \begin{cases} 1 & \mapsto 1^{t_1}2 \\ 2 & \mapsto 1^{t_2}3 \\ \vdots & \vdots \\ n+p-1 & \mapsto 1^{t_{n+p-1}}(n+p) \\ n+p & \mapsto 1^{t_{n+p}}(n+1). \end{cases}$$

Substitution associated with a simple Parry number — Substitution associated with a non-simple Parry number

We will use β-substitutions in Section 3 to describe properties of the set $\widetilde{\mathcal{R}_\beta}$. Note that the automaton $\mathcal{M}_\beta$ is exactly the prefix-suffix automaton of the substitution σ_β considered, for instance, in [CS01a, CS01b], after reversing all the edges and replacing the labels in the prefix-suffix automaton by the lengths of the prefixes.

The *incidence matrix* of the substitution σ_β is defined as the $d \times d$ matrix whose entry of index (i, j) counts the number of occurrences of the letter i in $\sigma(j)$. As a consequence of the definition, the incidence matrix of σ_β coincides with the transpose of the adjacency matrix of the automaton $\mathcal{M}_\beta$. By eigenvalue of a substitution σ, we mean in all that follows an eigenvalue of the characteristic polynomial of the incidence matrix of σ. A substitution is said to be of *Pisot type* if all its eigenvalues except its largest one, which is assumed to be simple, are non-zero and of modulus smaller than 1.

2. Representation of β-shifts

The right one-sided shift X_β^r admits the interval $[0, 1]$ as a natural geometric representation; namely, one associates with a sequence $(u_i)_{i\geq 1} \in X_\beta^r$ its real value $\sum_{i\geq 1} u_i\beta^{-i}$. We even have a measure-theoretical isomorphism between X_β^r endowed with the shift, and $[0, 1]$ endowed with the map T_β. We now want to give a similar geometric interpretation of the set X_β; we thus first give a geometric representation of X_β^l as a generalized Rauzy fractal.

2.1. Representation of the left one-sided shift X_β^l.

The aim of this section is first to introduce a formal power series, called formal representation of X_β^l, and secondly, a geometrical representation as an explicit compact set in the product of Euclidean and p-adic spaces following [Sie03]; the primes which appear as p-adic spaces here will be the prime factors of the norm of β.

Formal representation of the symbolic dynamical system X_β^l.

Definition 1. The *formal representation* of X_β^l is denoted by

$$\varphi_X : X_\beta \to \mathbb{Q}[[X]]$$

where $\mathbb{Q}[[X]]$ is the ring of formal power series with coefficients in $\mathbb{Q}$, and defined by:

$$\text{for all } (w_i)_{i\geq 0} \in X_\beta^l, \quad \varphi_X(w_i) = \sum_{i\geq 0} w_i X^i \in \mathbb{Q}[[X]].$$

Topologies over $\mathbb{Q}(\beta)$. We now want to specialize these formal power series by giving the value β to the indeterminate X, and associating with them values by making them converge. We thus want to find a topological framework in which all the series $\sum_{i\geq 0} w_i\beta^i$ for $(w_i)_{i\geq 0} \in X_\beta^l$ would converge; in fact, this boils down to finding all the Archemedean and non-Archemedean metrizable topologies on $\mathbb{Q}(\beta)$ for which these series converge in a suitable completion. They are of two types.

- Suppose that the topology is Archemedean. One of its associated equivalent absolute values $|\cdot|$ satisfies the following: its restriction to $\mathbb{Q}$ is given by the usual absolute value on $\mathbb{Q}$ and there exists a $\mathbb{Q}$-isomorphism τ_i such that $|x| = |\tau_i(x)|_{\mathbb{C}}$, for $x \in \mathbb{Q}(\beta)$. The series φ_X specialized in β converge in $\mathbb{C}$ if and only if τ_i is associated with a conjugate β_i of modulus strictly smaller than one.
- Assume that the topology is non-Archimedean: there exists a prime ideal $\mathcal{I}$ of the integer ring $\mathcal{O}_{\mathbb{Q}(\beta)}$ of $\mathbb{Q}(\beta)$ for which the topology coincides with the the $\mathcal{I}$-adic topology; let p be the prime number defined by $\mathcal{I} \cap \mathbb{Z} = p\mathbb{Z}$; the restriction of the topology to $\mathbb{Q}$ is the p-adic topology. The series φ_X specialized in β takes finite values in the completion $\mathbb{K}_\mathcal{I}$ of $\mathbb{Q}(\beta)$ for the $\mathcal{I}$-adic topology if and only if $\beta \in \mathcal{I}$, i.e., $|\beta|_\mathcal{I} < 1$.

Representation space $\mathbb{K}_\beta$ *of* X_β^l. We now assume that β is a Pisot number of say degree d. Let $\beta_2, \ldots, \beta_r$ be the real conjugates of β (they all have modulus strictly smaller than 1, since β is a Pisot number), and let β_{r+1}, $\overline{\beta_{r+1}}, \ldots, \beta_{r+s}, \overline{\beta_{r+s}}$ be its complex conjugates. For $2 \leq j \leq r$, let $\mathbb{K}_{\beta_j}$ be equal to $\mathbb{R}$, and for $r+1 \leq j \leq r+s$, let $\mathbb{K}_{\beta_j}$ be equal to $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{C}$ being endowed with the usual topology. For $i = 1$ to d, let τ_i be a $\mathbb{Q}$-automorphism of $\mathbb{K} = \mathbb{Q}(\beta_1, \ldots, \beta_d)$ which sends β on its algebraic conjugate β_i. For a given i and for every element $Q(\beta)$ of $\mathbb{Q}(\beta)$, then $\tau_i(Q(\beta)) = Q(\beta_i)$.

We first gather the complex representations by omitting those which are conjugate in the complex case. This representation contains all the possible Archimedean values for φ_X. It takes values in

$$\mathbb{K}_\infty = \mathbb{K}_{\beta_2} \times \cdots \times \mathbb{K}_{\beta_{r+s}}.$$

Let $\mathcal{I}_1, \ldots, \mathcal{I}_\nu$ be the prime ideals in the integer ring $\mathcal{O}_{\mathbb{Q}(\beta)}$ of $\mathbb{Q}(\beta)$ that contain β, that is,

$$\beta\mathcal{O}_{\mathbb{Q}(\beta)} = \prod_{i=1}^{\nu} \mathcal{I}_i^{n_i}. \tag{2.1}$$

Recall that $\mathbb{K}_{\mathcal{I}}$ stands for the completion of $\mathbb{Q}(\beta)$ for the $\mathcal{I}$-adic topology. We then gather the representations in the completions of $\mathbb{Q}(\beta)$ for the non-Archemedean topologies. Hence, one defines the *representation space of* X_β as the direct product $\mathbb{K}_\beta$ of all these fields:

$$\mathbb{K}_\beta = \mathbb{K}_{\beta_2} \times \dots \mathbb{K}_{\beta_{r+s}} \times \mathbb{K}_{\mathcal{I}_1} \times \dots \mathbb{K}_{\mathcal{I}_\nu} \simeq \mathbb{R}^{r-1} \times \mathbb{C}^s \times \mathbb{K}_{\mathcal{I}_1} \times \dots \times \mathbb{K}_{\mathcal{I}_\nu}.$$

The field $\mathbb{K}_{\mathcal{I}}$ is a finite extension of the $p_{\mathcal{I}}$-adic field $\mathbb{Q}_{p_{\mathcal{I}}}$ where $\mathcal{I} \cap \mathbb{Z} = p_{\mathcal{I}}\mathbb{Z}$. For a given prime p, the fields $\mathbb{K}_{\mathcal{I}}$ that are p-adic fields are those for which $\mathcal{I}$ simultaneously contains p and β. Furthermore, the prime numbers p for which there exists a prime ideal of $\mathcal{O}_{\mathbb{Q}(\beta)}$ which simultaneously contains p and β are exactly the prime divisors of the constant term of the minimal polynomial of β (see Lemma 4.2 [Sie03]). In particular, $\mathbb{K}_\beta$ is a Euclidean space if and only if β is a unit. The set $\mathbb{K}_\beta$ is a metric abelian group endowed with the product of the topologies of each of its elements,

The *canonical embedding* of $\mathbb{Q}(\beta)$ into $\mathbb{K}_\beta$ is defined by the following morphism:

$$\delta_\beta : P(\beta) \in \mathbb{Q}(\beta) \mapsto (\underbrace{P(\beta_2)}_{\in \mathbb{K}_{\beta_2}}, \dots, \underbrace{P(\beta_{r+s})}_{\in \mathbb{K}_{\beta_{r+s}}}, \underbrace{P(\beta)}_{\in \mathbb{K}_{\mathcal{I}_1}}, \dots, \underbrace{P(\beta)}_{\in \mathbb{K}_{\mathcal{I}_\nu}}) \in \mathbb{K}_\beta. \tag{2.2}$$

Since the topology on $\mathbb{K}_\beta$ has been chosen such that the formal power series $\lim_{n\to\infty} \delta_\beta(\sum_{i=0}^n w_i\beta^i) = \sum_{i\geq 0} w_i \delta_\beta(\beta)^i$ are convergent in $\mathbb{K}_\beta$ for every $(w_i)_{i\geq 0} \in X_\beta^l$, one defines the following, where the notation $\delta_\beta(\sum_{i\geq 0} w_i\beta^i)$ stands for $\sum_{i\geq 0} w_i\delta_\beta(\beta)^i$.

Definition 2. The *representation map of* X_β^l, called *one-sided representation map*, is defined by

$$\varphi_\beta : \ X_\beta^l \to \mathbb{K}_\beta, \ (w_i)_{i\geq 0} \mapsto \delta_\beta(\sum_{i\geq 0} w_i\beta^i).$$

We set $\mathcal{R}_\beta := \varphi_\beta(X_\beta^l)$ and call it the *generalized Rauzy fractal* or *geometric representation* of the left one-sided shift X_β^l.

2.2. **Examples.**

The golden ratio. Let $\beta = (1+\sqrt{5})/2$ be the golden ratio, i.e., the largest root of $X^2 - X - 1$. One has $d_\beta(1) = 11$ (β is a simple Parry number) and $d_\beta^*(1) = (10)^\infty$. Hence, X_β is the set of sequences in $\{0,1\}^{\mathbb{Z}}$ in which there are no two consecutive 1's. Furthermore, the associated β-substitution is the Fibonacci substitution: $\sigma_\beta : \ 1 \mapsto 12, \ 2 \mapsto 1$. One has $\mathbb{K}_\beta = \mathbb{R}$; the canonical embedding δ_β is reduced to the map $\tau_{(1-\sqrt{5})/2}$ (i.e., the $\mathbb{Q}$-automorphism of $\mathbb{Q}(\beta)$ which maps β on its conjugate), and $\delta_\beta(\mathbb{Q}(\beta)) = \mathbb{Q}(\beta)$. The set $\mathcal{R}_\beta$ is an interval.

The Tribonacci number. Let β be the *Tribonacci number*, i.e., the Pisot root of the polynomial $X^3 - X^2 - X - 1$. One has $d_\beta(1) = 111$ (β is a simple Parry number) and $d^*_\beta(1) = (110)^\infty$. Hence, X_β is the set of sequences in $\{0,1\}^{\mathbb{Z}}$ in which there are no three consecutive 1's. Furthermore, $\sigma_\beta :\ 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$. One has $\mathbb{K}_\beta = \mathbb{C}$; the canonical embedding is reduced to the $\mathbb{Q}$-isomorphism τ_α which maps β on α, where α is one of the complex roots of $X^3 - X^2 - X - 1$. The set $\mathcal{R}_\beta$ which satisfies

$$\mathcal{R}_\beta = \{\sum_{i \geq 0} w_i \alpha^i;\ \forall i,\ w_i \in \{0,1\},\ w_i w_{i+1} w_{i+2} \neq 0\}$$

is a compact subset of $\mathbb{C}$ called the *Rauzy fractal.* This set was introduced in [Rau82], see also [IK91, Mes98, Mes00]. This is shown in Fig. 2.2 with a division into three pieces indicated by different shades. They correspond to the sequences $(w_i)_{i\geq 0}$ such that either $w_0 = 0$, or $w_0w_1 = 10$, or $w_0w_1 = 11$. There are as many pieces as the length of $d_\beta(1)$, which is also equal here to the degree of β. We will come back to the advantages of this division of Rauzy fractals into smaller pieces in Section 3.

The smallest Pisot number. Let β be the Pisot root of $X^3 - X - 1$. One has $d_\beta(1) = 10001$ (β is a simple Parry number) and $d^*_\beta(1) = (10000)^\infty$; $\sigma_\beta :\ 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 4,\ 4 \mapsto 5,\ 5 \mapsto 1$; the characteristic polynomial of its incidence matrix is $(X^3 - X - 1)(X^2 - X + 1)$, hence σ_β is not a Pisot type substitution. A self-similar tiling generated by it has been studied in detail in [AS98]; some connected surprising tilings have also been introduced in [EI05]. One has $\mathbb{K}_\beta = \mathbb{C}$; the canonical embedding is also reduced to the $\mathbb{Q}$-isomorphism τ_α which maps β on α, where α is one of the complex roots of $X^3 - X - 1$. The set $\mathcal{R}_\beta$ is shown in Fig. 2.2 with a division into five pieces respectively corresponding to the sequences $(w_i)_{i\geq 0}$ such that either $w_0w_1w_2w_3 = 0000$, or $w_0 = 1$, $w_0w_1 = 01$, $w_0w_1w_2 = 001$, or $w_0w_1w_2w_3 = 0001$. The number of different pieces is equal to the length of $d_\beta(1)$; there are here 5 pieces whereas the degree of β is 3.

A non-unit example. Let $\beta = 2+\sqrt{2}$ be the dominant root of the polynomial $X^2 - 4X + 2$. The other root is $2 - \sqrt{2}$. One has $d_\beta(1) = d^*_\beta(1) = 31^\infty$; β is not a simple Parry number; $\sigma_\beta :\ 1 \mapsto 1112,\ 2 \mapsto 12$. The ideal $2\mathbb{Z}$ is ramified in $\mathcal{O}_{\mathbb{Q}(\sqrt{2})}$, i.e., $2\mathbb{Z} = \mathcal{I}^2$. Hence there exists only one ideal which contains $\sqrt{2}$; its ramification index is 2; the degree of the extension $\mathbb{K}_{\mathcal{I}}$ over $\mathbb{Q}_2$ has degree 2. Hence the geometric one-sided representation $\mathcal{R}_\beta$ is a subset of $\mathbb{R} \times \mathbb{Q}_2 \times \mathbb{Q}_2$, shown in Fig. 2.2. The division of the Rauzy fractal cannot be expressed as in the previous examples as finite conditions on prefixes of the sequences $(w_i)_{i\geq 0}$ (for more details, see Section 3.3).

2.3. **Representation of the two-sided shift** (X_β, S)**.**

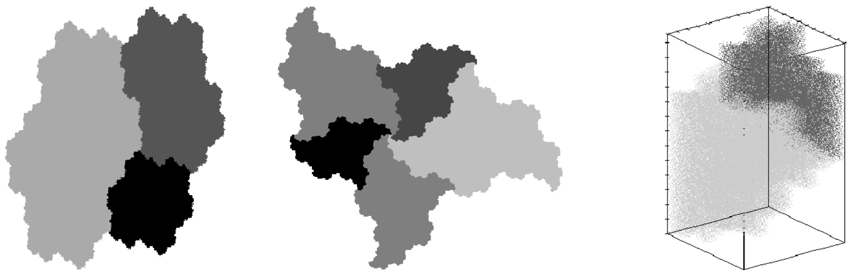

FIGURE 2.1. The Rauzy geometric one-sided representation for the Tribonacci-shift, the smallest Pisot number-shift and the $(2+\sqrt{2})$-shift.

Representation space $\widetilde{\mathbb{K}_\beta}$ *of the two-sided* β*-shift* X_β *and representation map* $\widetilde{\varphi_\beta}$. We now define the *representation map* $\widetilde{\varphi_\beta}$ of X_β: it takes its values in $\mathbb{K}_\beta \times \mathbb{R}$ and maps a point $(w,u) = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1})$ in X_β to the point obtained by introducing, on the last coordinate (in $\mathbb{R}$), the real number whose β-expansion is given by $(u_i)_{i\geq 1}$, and by gathering, on the first coordinate (in $\mathbb{K}_\beta$), the set of finite values which can be taken by the formal power series $\varphi_X(w)$ specialized in β for all the topologies that exist on $\mathbb{Q}(\beta)$. Let us define $\widetilde{\mathbb{K}_\beta}$ as $\mathbb{K}_\beta \times \mathbb{R}$.

Definition 3. The *representation map* $\widetilde{\varphi_\beta} : X_\beta \to \widetilde{\mathbb{K}_\beta}$ *of* X_β, called *two-sided representation map*, is defined for all $(w,u) = ((w_i)_{i\geq 0}, (u_i)_{i\geq 1}) \in X_\beta$ by:

$$\widetilde{\varphi_\beta}(w,u) = (-\varphi_\beta((w_i)_{i\geq 0}), \sum_{i\geq 1} u_i\beta^{-i}) = (-\delta_\beta(\sum_{i\geq 0} w_i\beta^i), \sum_{i\geq 1} u_i\beta^{-i}) \in \mathbb{K}_\beta \times \mathbb{R}.$$

The set $\widetilde{\mathcal{R}_\beta} := \widetilde{\varphi_\beta}(X_\beta)$ is called the *Rauzy fractal* or *geometric representation* of the two-sided β-shift. This set is easily seen to be bounded and hence compact.

We will see in the first part of the proof of Theorem 3 the advantages of introducing the sign minus before δ_β in the definition of the map $\widetilde{\varphi_\beta}$.

Examples. The geometric representation map $\widetilde{\varphi_\beta}$ of the golden ratio-shift maps to $\mathbb{R}^2$. Those for the Tribonacci number and the smallest Pisot number map to $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. They are shown (up to a change of coordinates) in Fig. 2.2. The sets $\widetilde{\mathcal{R}_\beta}$ are unions of products of the different pieces of the Rauzy fractal $\mathcal{R}_\beta$ by finite real intervals of different heights. For instance, in the Tribonacci case, since the different pieces in $\mathcal{R}_\beta$ correspond to the sequences $(w_i)_{i\geq 0}$ such that either $w_0 = 0$, or $w_0w_1 = 10$, or $w_0w_1 = 11$, this gives different constraints on the sequences $(u_i)_{i\geq 1}$ which produce the real component; for instance, the piece which corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 11$ implies the following constraint on the sequences $(u_i)_{i\geq 1}$: u_0 has to be equal to 0.

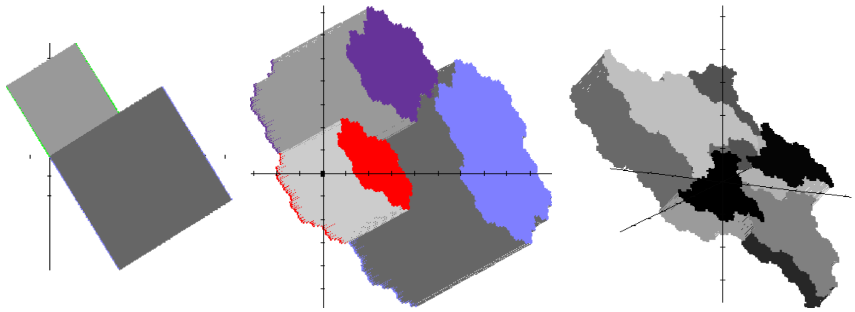

FIGURE 2.2. The geometric representation of the two-sided β-shift for the Fibonacci-shift, the Tribonacci-shift, and the smallest Pisot number-shift.

Remark 1. The definition of the representation map $\widetilde{\varphi_\beta}$ is inspired by [Sie03] in the substitutive case. We cannot directly apply here the substitutive formalism to the substitution σ_β since it is generally not a Pisot number: the dominant eigenvalue of σ_β is still a Pisot number but other eigenvalues may occur, as in the smallest Pisot case. The main difference here is that we do not take these extra eigenvalues into account in our definition of $\widetilde{\varphi_\beta}$.

3. GEOMETRIC PROPERTIES OF THE RAUZY FRACTAL OF THE β-NUMERATION

The aim of this section is to prove some geometric properties of the Rauzy fractal associated with the β-numeration:

Theorem 2. *Let β be a Pisot number.*

(1) *The Rauzy fractal $\mathcal{R}_\beta$ of the one-sided β-shift has a graph directed self-affine structure in the sense of* [MW88]*. More precisely:*
   - *it has a non-zero measure for the Haar measure;*
   - *there are d pieces which form the self-affine structure, where d is equal to the period of $d_\beta(1)^*$ if β is a simple Parry number or otherwise to the sum of its preperiod plus its period.*

(2) *The Rauzy fractal $\widetilde{\mathcal{R}_\beta}$ of the two-sided β-shift has non-zero measure for the Haar measure $\mu_{\mathbb{K}_\beta}$ over $\mathbb{K}_\beta \times \mathbb{R}$. It is the disjoint union of d cylinders obtained as the product of each piece of the one-sided Rauzy fractal by a finite interval of $\mathbb{R}$.*

(3) *The one-sided (resp. two-sided) representation map φ_β (resp. $\widetilde{\varphi_\beta}$) is one-to-one except on a set of zero measure.*

(4) *Let σ_β be the β-substitution associated with β such as defined in Section 1. We assume that the characteristic polynomial of the incidence matrix of σ_β is irreducible, i.e., β is a simple Parry number and the length of $d_\beta(1)$ equals the degree of β. Let $\mathbf{v}$ be the left Perron-Frobenius eigenvector associated with the dominant eigenvalue of the incidence matrix of σ_β normalized such that its first*

coordinate equals 1*. Let* $\mathcal{G}$ *be the compact quotient group obtained by factorizing the closed subgroup of* $\mathbb{K}_\beta$ *generated by the vectors* $\delta_\beta(\mathbf{v}_i)$*, for* $i = 2, \cdots, d$*, by the discrete subgroup generated by the vectors* $\delta_\beta(\mathbf{v}_i) - \delta_\beta(\mathbf{v}_1)$*, for* $i = 2, \cdots, d$*. Let* π *be the continuous projection onto* $\mathcal{G}$*. There exists a positive integer* k *such that the map* $\pi \circ \varphi_\beta$ *is almost everywhere* k*-to-one.*

Similarly, there exists a compact quotient group $\widetilde{\mathcal{G}}$ *of a closed subgroup of* $\widetilde{\mathbb{K}_\beta}$ *such that the projection* $\widetilde{\pi}$ *onto* $\widetilde{\mathcal{G}}$ *is almost everywhere constant-to-one.*

3.1. **Proof of Theorem 2.** The proof of the first point of this theorem is divided into several steps. Roughly, we use the self-affine structure of $\mathcal{R}_\beta$ to deduce that it has non-zero measure. Similar statements are proved in the framework of Pisot substitutions in [Sie03], but here we cannot directly use these statements since the β-substitutions (Section 1) used here are not necessarily of Pisot type according to Remark 1, as for instance in the smallest Pisot number case.

Division of $\mathcal{R}_\beta$ into subpieces. Let us first divide (as illustrated in Figure 2.2) the set $\mathcal{R}_\beta$ into d pieces, where d stands for the number of states in the minimal automaton $\mathcal{M}_\beta$ which recognizes the set of factors $F(X_\beta)$ of the two-sided shift X_β.

Let $\widetilde{\mathcal{M}_\beta}$ denote the (non-deterministic) automaton obtained from $\mathcal{M}_\beta$ by reversing the orientation of the edges. The set of sequences in the left one-sided shift X_β^l is equal to the set of labels of infinite one-sided paths in the automaton $\widetilde{\mathcal{M}_\beta}$. We label the states of the automaton $\widetilde{\mathcal{M}_\beta}$ by $a_1, \cdots, a_d$. For $i = 1, \cdots, d$, one defines

$$\mathcal{R}_\beta(i) := \left\{ \begin{array}{c} \varphi_\beta((w_k)_{k\geq 0});\ (w_k)_{k\geq 0} \in \mathcal{A}_\beta^{\mathbb{N}^*}; (w_k)_{k\geq 0} \\ \text{is a path from the state } a_i \text{ in } \widetilde{\mathcal{M}_\beta} \end{array} \right\}.$$

Let us recall that d is larger than or equal to the degree r of the minimal polynomial of β. For instance, $r = 3$ and $d = 5$ in the smallest Pisot case. In particular, d can be arbitrarily large when β is cubic, according to [Bas02].

Self-affine decomposition. Let the map $h_\beta : \mathbb{K}_\beta \to \mathbb{K}_\beta$ stand for the multiplication map in $\mathbb{K}_\beta$ by the diagonal matrix whose diagonal coefficients are given by $\delta_\beta(\beta)$. Hence, $\delta_\beta(\beta x) = h_\beta \delta_\beta(x)$ for every $x \in \mathbb{Q}(\beta)$.

Let us first prove that for $i = 1, \cdots, d$:

$$\mathcal{R}_\beta(i) = \cup_{1\leq j\leq d} \cup_{p,\ \sigma_\beta(j)=pis}\, h_\beta(\mathcal{R}_\beta(j)) + \delta_\beta(|p|). \tag{3.1}$$

Let $i \in \{1, \cdots, d\}$ be given. Let $(w_k)_{k\geq 0}$ be a path from the state a_i. Then:

$$\begin{aligned} \varphi_\beta((w_k)_{k\geq 0}) &= \delta_\beta(\textstyle\sum_{k\geq 1} w_k\beta^k) + \delta_\beta(w_0) = h_\beta\delta_\beta(\sum_{k\geq 1} w_k\beta^{k-1}) + \delta_\beta(w_0) \\ &= h_\beta\varphi_\beta((w_k)_{k\geq 1}) + \delta_\beta(w_0). \end{aligned}$$

Let a_j be the state in $\widetilde{\mathcal{M}_\beta}$ obtained by reading the label w_0 from the state a_i. By the definition of σ_β, there exist $p = 1^{w_0}$ and s such that $\sigma_\beta(j) = pis$, that is, $w_0 = |p|$. Hence

$$\varphi_\beta((w_k)_{k\geq 0}) \in h_\beta(\mathcal{R}_\beta(j)) + \delta_\beta(|p|),$$

which provides one inclusion for the equality (3.1). The other inclusion is then immediate.

The Rauzy fractal has non-zero measure. The proof is an adaption of the corresponding proof of [Sie03] which is done in the framework of Pisot substitution dynamical systems; two properties are recalled below without a proof, the first one describes the action of the multiplication map h_β on the Haar measure $\mu_{\mathbb{K}_\beta}$ of $\mathbb{K}_\beta$:

Lemma 1 ([Sie03])**.** (1) *For every Borelian set B of $\mathbb{K}_\beta$*

$$\mu_{\mathbb{K}_\beta}(h_\beta(B)) = \frac{1}{\beta}\mu_{\mathbb{K}_\beta}(B).$$

(2) *Let $\mathcal{S}$ be a finite set included in $\mathbb{Q}(\beta)$. The set of points $\{\delta_\beta(P(1/\beta));\ P \in \mathcal{S}[X]\}$ is a uniformly discrete set in $\mathbb{K}_\beta$.*

Let $(U_N)_{N\in\mathbb{N}}$ be the linear canonical numeration system associated with β according to [BM89], i.e., $U_0 = 1$, and for all k, $U_k = t_1U_{k-1} + \cdots + t_kU_0 + 1$, where $d^*_\beta(1) = (t_k)_{k\in\mathbb{N}}$. Let us expand every integer $i = 0, 1, \cdots, U_N - 1$ in this system according to the greedy algorithm (with this expansion being unique):

$$i = \sum_{0\leq k\leq N-1} w_k^{(i)} U_k.$$

According to [BM89] (see also [Fro02]), the finite words $w_{N-1}^{(i)} \cdots w_0^{(i)}$, for $i = 0, 1, \cdots, U_N - 1$ are all distinct and all belong to the set $F(X_\beta)$ of factors of elements of X_β. Hence, the sequences $(w_0^{(i)} \cdots w_{N-1}^{(i)})0^\infty)$, for $i = 0, 1, \cdots, U_N - 1$, all belong to the left one-sided β-shift X^l_β.

Let $\mathcal{E}_N$ be the image under the action of φ_β of this set of points. One has for $i = 0, 1, \cdots, U_N - 1$,

$$\varphi_\beta(w_0^{(i)} \cdots w_{N-1}^{(i)})0^\infty) = \delta_\beta(\sum_{0\leq k\leq N-1} w_k^{(i)}\beta^i).$$

The points in $\mathcal{E}_N$ are all distinct since for $i = 0, 1, \cdots, U_N - 1$, the finite words $w_{N-1}^{(i)} \cdots w_0^{(i)}$ are all distinct. There thus exists $B > 0$ such that $\mathrm{Card}\, \mathcal{E}_N = U_N > B\beta^N$, since β is a Pisot number.

Let us now apply Lemma 1 with $\mathcal{S} = \{0, 1, \cdots, [\beta]\}$ to the set $(h_\beta)^{-N}\mathcal{E}_N$, i.e., to the set of points $\delta_\beta(\sum_{0\leq k\leq N-1} w_k^{(i)}\beta^{i-N})$, for $i = 0, 1, \cdots, U_N - 1$. There exists a constant $A > 0$ such that the distance between two elements of $\mathcal{E}_N$ is larger than A. Let us now define for every non-negative integer N, the set

$$\mathcal{B}_N = \cup_{z\in\mathcal{E}_N}(h_\beta)^N B(z, A/2),$$

where $B(z, A/2)$ denotes the closed ball in $\mathbb{K}_\beta$ of center z and radius $A/2$. According to Lemma 1 and to the fact that Card $\mathcal{E}_N > B\beta^N$, for all N, there exists $C > 0$ such that $\mu_{\mathbb{K}_\beta}(\mathcal{B}_N) > C$, for all N.

The main point now is that the sequence of compact sets $(\mathcal{B}_N)_{N\in\mathbb{N}}$ converges toward a subset of $\mathcal{R}_\beta$ with respect to the Hausdorff metric. Indeed, for a fixed positive integer p, when N is large enough, $\mathcal{B}_N \subset \mathcal{R}_\beta(1/p) := \{x \in \mathbb{K}_\beta;\ d(x, \mathcal{R}_\beta) \leq 1/p\}$. Independently, fix $\varepsilon > 0$; since the sequence $(\mathcal{R}_\beta(1/p))_p$ converges toward $\mathcal{R}_\beta$, there exists $p > 0$ such that $\mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta(1/p)) \leq \mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta) + \varepsilon$. This finally implies $\liminf \mu_{\mathbb{K}_\beta}(\mathcal{B}_N) \leq \mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta) + \varepsilon$. Since this holds for every $\varepsilon > 0$, one obtains $\mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta) \geq C > 0$.

Computation of measures and self-affine structure. Let us now prove that the union in (3.1) is a disjoint union up to sets of zero measure. Then for a given $i \in \{1, \cdots, d\}$ according to (3.1) and to Lemma 1

$$\begin{aligned} \mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta(i)) &\leq \textstyle\sum_{j=1,\cdots,d,\ \sigma_\beta(j)=pis} \mu_{\mathbb{K}_\beta}(h_\beta(\mathcal{R}_\beta(j))) \\ &\leq 1/\beta \textstyle\sum_{j=1,\cdots,d,\ \sigma_\beta(j)=pis} \mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta(j)). \end{aligned} \tag{3.2}$$

Let $\mathbf{m} = (\mu_{\mathbb{K}_\beta}(\mathcal{R}_\beta(i)))_{i=1,\cdots,d}$ stand for the vector in $\mathbb{R}^d$ of measures in $\mathbb{K}_\beta$ of the pieces of the Rauzy fractal. We know from what precedes that $\mathbf{m}$ is a non-zero vector with non-negative entries. According to the Perron-Frobenius theorem, the previous equality implies that $\mathbf{m}$ is an eigenvector of the primitive incidence matrix of the substitution σ_β. We thus have equality in (3.2), which implies that the unions are disjoint up to sets of zero measure. One similarly proves that this equality in measure still holds by replacing σ by σ^n for every n.

Now take two distinct pieces, say $(\mathcal{R}_\beta(j))$ and $(\mathcal{R}_\beta(k))$, with $j \neq k$. There exists n such that both $\sigma^n(j)$ and $\sigma^n(k)$ admit as first letter 1. Hence, they both occur in (3.1) for $i = 1$ with the same translation term (which is indeed equal to 0) and they are thus distinct. We have therefore proved that the d pieces of the Rauzy fractal $(\mathcal{R}_\beta(j))$ are disjoint up to sets of zero measure, which ends the proof of Assertion (i), i.e., $\mathcal{R}_\beta$ has a self-affine structure, which ends the proof of (1).

$\widetilde{\mathcal{R}_\beta}$ has non-zero measure. Let us prove Assertion (2) of the theorem. The assertion that $\widetilde{\mathcal{R}_\beta}$ has non-zero measure is a direct consequence of the structure of the Rauzy fractal $\widetilde{\mathcal{R}_\beta}$ studied above since $\widetilde{\mathcal{R}_\beta}$ can be decomposed as the disjoint (in measure) union of $\widetilde{\mathcal{R}_\beta}(i)$, $i = 1, \cdots, d$, where

$$\widetilde{\mathcal{R}_\beta}(i) := \left\{ \begin{array}{l} \widetilde{\varphi_\beta}(u,w);\ (u,w) \text{ is a two-sided path in } \widetilde{\mathcal{M}_\beta} \\ \text{such that } w \text{ starts from the state } a_i \end{array} \right\},$$

which is the product of $\mathcal{R}_\beta(i)$ by a finite interval of $\mathbb{R}$ of non-zero measure.

Proof of (3). We deduce that φ_β is one-to-one from (3.1) and from the fact that the d pieces of the Rauzy fractal $(\mathcal{R}_\beta(j))$ are disjoint up to sets of zero measure. For more details, see e.g. [CS01b, Sie03]. We similarly deduce the same result for $\widetilde{\varphi_\beta}$.

Proof of (4). We follow here again [Sie03]. The Rauzy fractal $\mathcal{R}_\beta$ is included in the closed subgroup of $\mathbb{K}_\beta$ generated by the vector $\delta_\beta(\mathbf{v}_1)$. The subgroup generated by the vectors $\delta_\beta(\mathbf{v}_i) - \delta_\beta(\mathbf{v}_1)$, for $i = 2, \cdots, d$, is discrete since its projection in $\mathbb{R}^{d-1}$ is discrete, according to [CS01b] (we use here the irreducibility assumption on σ_β). The projection of $\mathcal{R}_\beta$ onto $\mathcal{G}$ has finite fibers by compacity of $\mathcal{R}_\beta$. We deduce from (3) that $\pi \circ \varphi_\beta$ is finite-to-one almost everywhere. We can go even further: the fibers of the quotient map are constant almost everywhere, by following the same arguments as B. Host's proof in the unimodular case, which is detailed in Exercise 7.5.14 in [Fog02]. Indeed, let T stand for the left-sided odometer acting on X_β^l induced by the adic transformation defined on the Markov compactum [Ver81] provided by the automaton $\widetilde{\mathcal{M}_\beta}$ (map T is topologically isomorphic to the the shift acting on the two-sided symbolic dynamical system generated by the substitution σ_β); the map that associates with a point w in X_β^l the cardinality of $\pi \circ \varphi_\beta^{-1}(w)$ is measurable and invariant under the action of the ergodic transformation T, hence it is constant almost everywhere. Again we similarly deduce the same result for $\widetilde{\varphi_\beta}$.

3.2. **Remarks.** Similar to what is described in [Sch00], our construction also works for any algebraic number whose conjugates have absolute values distinct from 1. Indeed, we still have a candidate for the symbolic coding representing an hyperbolic automorphism in the non-Pisot case by introducing, in the representation map $\widetilde{\varphi_\beta}$, the coordinate $\sum_{i\geq 1} u_i\lambda^{-i}$ for conjugates λ of modulus strictly larger that 1 (see for instance [KV98] for a similar description). Yet we lose all the geometric properties of Theorem 2. We do not know, for instance, if Rauzy fractals still have zero measure, with the Salem case being even worse since we may lose convergence in our formal series.

More precisely, we are not able to prove that the union in Equation (3.1) is disjoint up to sets of zero measure and that the Haar measure of the generalized Rauzy fractal is non-zero. Indeed, the disjointness in (3.1) requires two arguments: first, Inequality (3.2) which holds in the Salem case, and secondly, the pieces of the Rauzy fractal have nonzero measure. Concerning this latter point, our proof requires the Pisot hypothesis. In other words, the disjointness in (3.1) is equivalent to the fact that $\widetilde{\varphi_\beta}$ is one-to-one onto the generalized Rauzy fractal. The fact that $\widetilde{\varphi_\beta}$ factorizes as a one-to-one map onto the group $\widetilde{\mathcal{G}}$ of (4) is much more difficult, and corresponds to the problem of the essential injectivity discussed in [Sch00], or to the so-called *Pisot conjecture* (e.g. see [BK06, BS05, Sie03]).

3.3. **Examples.** Let us pursue the study of three of the examples of Section 2.2.

The Tribonacci number. Let us recall that

$$\mathcal{R}_\beta = \{\sum_{i\geq 0} w_i\alpha^i;\ \forall i,\ w_i \in \{0,1\},\ w_iw_{i+1}w_{i+2} \neq 0\}.$$

One easily checks, thanks to the automaton $\widetilde{\mathcal{M}_\beta}$ shown in Fig. 3.1, that $\mathcal{R}_\beta(1)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0 = 1$, $\mathcal{R}_\beta(2)$ corresponds to the set of sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 10$, and lastly, $\mathcal{R}_\beta(3)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 11$. One has

$$\begin{cases} \mathcal{R}_\beta(1) = \alpha(\mathcal{R}_\beta(1) \cup \mathcal{R}_\beta(2) \cup \mathcal{R}_\beta(3)) \\ \mathcal{R}_\beta(2) = \alpha(\mathcal{R}_\beta(1)) + 1 \\ \mathcal{R}_\beta(3) = \alpha(\mathcal{R}_\beta(2)) + 1. \end{cases}$$

The smallest Pisot number. One has

$$\begin{array}{rl} \mathcal{R}_\beta = & \{\sum_{i\geq 0} w_i\alpha^i;\ \forall i,\ w_i \in \{0,1\},\quad \text{if } w_i = 1, \\ & \text{then } w_{i+1} = w_{i+2} = w_{i+3} = w_{i+4} = 0\}. \end{array}$$

One easily checks that $\mathcal{R}_\beta(1)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0 = 0000$, $\mathcal{R}_\beta(2)$ corresponds to the set of sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 1$, $\mathcal{R}_\beta(3)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 01$, $\mathcal{R}_\beta(4)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 001$, and lastly $\mathcal{R}_\beta(5)$ corresponds to the sequences $(w_i)_{i\geq 0}$ such that $w_0w_1 = 0001$.

One has

$$\begin{cases} \mathcal{R}_\beta(1) = \alpha(\mathcal{R}_\beta(1) \cup \mathcal{R}_\beta(5)) \\ \mathcal{R}_\beta(2) = \alpha(\mathcal{R}_\beta(1)) + 1 \\ \mathcal{R}_\beta(3) = \alpha(\mathcal{R}_\beta(2)) \\ \mathcal{R}_\beta(4) = \alpha(\mathcal{R}_\beta(3)) \\ \mathcal{R}_\beta(5) = \alpha(\mathcal{R}_\beta(4)). \end{cases}$$

The $(2+\sqrt{2})$-shift. One has

$$\mathcal{R}_\beta = \{\sum_{i\geq 0} w_i(2-\sqrt{2})^i;\ \forall i,\ w_i \in \{0,1,2\},\ (w_i)_{i\geq 0} \leq_{\text{lex}} (31^\infty)\}.$$

In this non-simple Parry case, we cannot express the sets $\mathcal{R}_\beta(1)$ and $\mathcal{R}_\beta(2)$ as easily as above: one checks in Figure 3.1 that there exist cycles from a_1 to a_1 and from a_2 to a_2, which implies that both $\mathcal{R}_\beta(1)$ and $\mathcal{R}_\beta(2)$ contain sequences with arbitrarily long common prefixes, such as 1^n for every n.

One has

$$\begin{cases} \mathcal{R}_\beta(1) = & (2-\sqrt{2})(\mathcal{R}_\beta(1)) \cup ((2-\sqrt{2})(\mathcal{R}_\beta(1)) + 1) \cup ((2-\sqrt{2})(\mathcal{R}_\beta(1)) + 2) \\ & \cup (2-\sqrt{2})(\mathcal{R}_\beta(2)) \\ \mathcal{R}_\beta(2) = & (2-\sqrt{2})(\mathcal{R}_\beta(1)) + 3 + (2-\sqrt{2})(\mathcal{R}_\beta(2)) + 1. \end{cases}$$

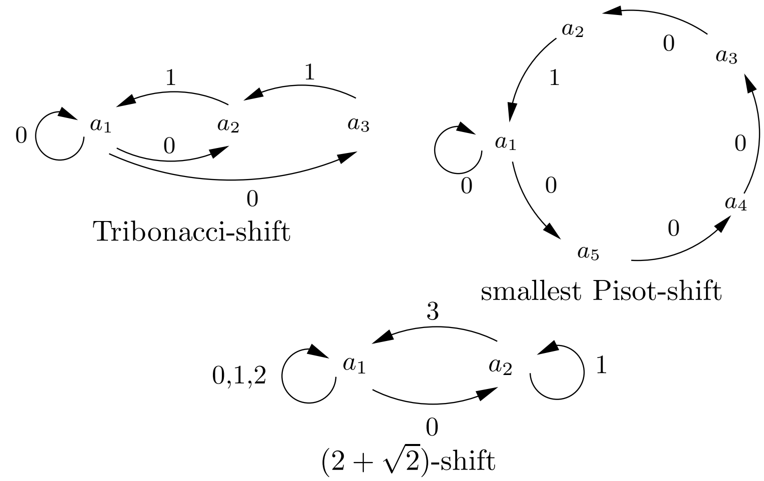

FIGURE 3.1. Reversed minimal automaton $\widetilde{\mathcal{M}_\beta}$ describing the structure of the β-shift.

4. Characterization of purely periodic points

We can now state the main theorem of this paper.

Theorem 3. *Let β be a Pisot number.*

- *Assume that β is a non-simple Parry number. For all $z \in \mathbb{Q}(\beta) \cap [0,1)$, the β-expansion of z is purely periodic if and only if $(\delta_\beta(z), z) \in \widetilde{\mathcal{R}_\beta} = \widetilde{\varphi_\beta}(X_\beta)$.*
- *Assume that β is a simple Parry number. For all $z \in \mathbb{Q}(\beta) \cap [0,1)$, the β-expansion of z is purely periodic or there exists $k \in \mathbb{N}^*$ such that $z = T_\beta^k(1)$ if and only if $(\delta_\beta(z), z) \in \widetilde{\mathcal{R}_\beta} = \widetilde{\varphi_\beta}(X_\beta)$.*

Proof

Let us assume that the β-expansion of z is purely periodic. Write z as $z = 0.(a_1 \dots a_L)^\infty$, and set $w = {}^\infty(a_1 \dots a_L)$ and $u = (a_1 \dots a_L)^\infty$ (i.e., $w = (w_i)_{i\geq 0}$ with $w_0 \cdots w_{L-1} = a_L \dots a_1$, and $w_{i+L} = w_i$ for all i, and $u = (u_i)_{i\geq 1}$, with $u_1 \cdots u_L = a_1 \dots a_L$, and $u_{i+L} = u_i$ for all i). Then $(w,u) \in X_\beta$ acording to (1.1). Let us compute $\widetilde{\varphi_\beta}(w,u)$. Note that the second coordinate of $\widetilde{\varphi_\beta}(w,u)$ is z:

$$\sum_{i\geq 1} u_i \beta^{-i} = \frac{a_1\beta^{-1} + \cdots + a_L\beta^{-L}}{1-\beta^{-L}} = \frac{a_1\beta^{L-1} + \cdots + a_L}{\beta^L - 1} = z.$$

Futhermore, $\lim_{n\to\infty}\delta_\beta(\beta^n)=0$ in $\mathbb{K}_\beta$. We thus have

$$\begin{aligned}
\widetilde{\varphi_\beta}(w,u) &= \left(-\delta_\beta(\sum_{i\geq 0} w_i\beta^i), \sum_{i\geq 1} u_i\beta^{-i}\right)\\
&= \left(-\lim_{n\to\infty}\delta_\beta((a_L+\cdots+a_1\beta^{L-1})(1+\beta^L+\cdots+\beta^{nL})), z\right)\\
&= \left(\lim_{n\to\infty}\delta_\beta\left(-(a_1\beta^{L-1}+\cdots+a_L)\frac{1-\beta^{nL}}{1-\beta^L}\right), z\right)\\
&= \left(\delta_\beta\left(\frac{a_1\beta^{L-1}+\cdots+a_L}{\beta^L-1}\right), z\right)\\
&= (\delta_\beta(z), z),
\end{aligned}$$

hence $(\delta_\beta(z),z)\in\widetilde{\mathcal{R}_\beta}$.

We similarly prove that if β is a simple Parry numbe, and if there exists $k\in\mathbb{N}^*$ such that $z=T_\beta^k(1)$, then $(\delta_\beta(z),z)\in\widetilde{\mathcal{R}_\beta}=\widetilde{\varphi_\beta}(X_\beta)$.

Consider now the converse and let $z\in\mathbb{Q}(\beta)\cap[0,1)$ such that $(\delta_\beta(z),z)\in\widetilde{\mathcal{R}_\beta}$. We furthermore assume that $z\neq 0$ (if $z=0$, then its β-expansion is purely periodic). There exists $(w,u)=((w_i)_{i\geq 0},(u_i)_{i\geq 1})\in X_\beta$ such that $\widetilde{\varphi_\beta}(w,u)=(\delta_\beta(z),z)$. Consequently, one has $z=\sum_{i\geq 1}u_i\beta^{-i}$. We want to prove that either the sequence $(u_i)_{i\geq 1}$ satisfies (1.1) (it is the β-expansion of z) or else, that β is a simple Parry number, and the sequence $(u_i)_{i\geq 1}=S^k(d_\beta^*(1))$, for some $k\in\mathbb{N}$. In both cases, we will deduce that the sequence $(u_i)_{i\geq 1}$ is purely periodic.

The sketch of the proof inspired partly by [Sch80] and partly by [IR05], with both papers dealing with the unit case. In the original proof of [IR05], the analogous of the finite set $\mathcal{S}_z$ that we introduce below is $\mathbb{Z}[\beta]/q$ for an integer q which depends on z, with this set being no longer stable under the multiplication by $1/\beta$ in the non-unit case.

Let us define a sequence of points $(z_k)_{k\in\mathbb{N}}$ with values in $\mathbb{Q}(\beta)$ as follows: we set $z_0=z$, and $z_k=\frac{z+\sum_{0\leq i<k}w_i\beta^i}{\beta^k}\in\mathbb{Q}(\beta)$ for $k>0$. Let us prove that the set of points $\mathcal{S}_z=\{z_k;k\in\mathbb{N}\}$ is finite by stating that it is uniformly bounded for all $\mathcal{I}$-adic topologies which correspond to prime ideals $\mathcal{I}$ which do not appear in the decomposition (2.1). Indeed, as already stated, from the Pisot assumption, there exists exactly one topology such that $|\beta|>1$, i.e., the usual topology on $\mathbb{Q}(\beta)$. All other topologies such that $|\beta|<1$ are taken into account in the constuction of δ_β.

- Let $|\cdot|$ stand for the usual Archimedean absolute value on $\mathbb{Q}(\beta)$. Since
$$z_k=\sum_{0\leq i<k}w_k\beta^{i-k}+\sum_{i\geq 1}u_i\beta^{-k-i},$$
and according to (1.2), then $0\leq z_k\leq 1$, for all $k\in\mathbb{N}$.

- Let $|\cdot|$ be a metric on $\mathbb{Q}(\beta)$ such that $|\beta| < 1$. Then the completion of $\mathbb{Q}(\beta)$ for this topology appears as a coordinate of the embedding map δ_β. From $\delta_\beta(\sum_{i\geq 0} w_i\beta^i) = -\delta_\beta(z)$, we deduce that $\sum_{i\geq k} w_i\beta^{i-k}$ tends to z_k for this topology. Hence, $|z_k| \leq [\beta]\sum_{i\geq 0}|\beta|^i = \frac{[\beta]}{1-|\beta|}$.

Furthermore, there exists $q \in \mathbb{Z}[\beta]$ such that the set of points $\mathcal{S}_z = \{z_k; k \in \mathbb{N}\}$ is included in $\mathbb{Z}[\beta]/q$ and the norms of its elements are uniformly bounded. We thus deduce that it is finite.

Let us explicitly describe the restriction of the β-transformation T_β on $\mathcal{S}_z$. By construction, we have $\beta z_{k+1} = w_k + z_k$, for every $k \in \mathbb{N}$. Let $k \geq 1$. One has $z_k \in [0,1]$. We distinguish two cases:

- If $z_k < 1$, then one has $T_\beta(z_{k+1}) = z_k$.
- If $z_k = 1$, then $w_{k-1}\dots w_0u_1\dots u_n\cdots = d_\beta^*(1)$, and $T_\beta(z_{k+1}) = \{\beta z_{k+1}\} = \beta z_k + 1 - w_k - 1 = 0$.

We now have to consider the three following possibilities, acording to the number of times that the sequence $(z_k)_{k\in\mathbb{N}}$ takes the value 1:

(1) We first suppose that $z_k \neq 1$ for all $k \in \mathbb{N}$. Since the set $\mathcal{S}_z$ is finite, there exist two indices $0 \leq i < j$ such that $z_i = z_j$. From above, we know that $z_0 = T_\beta^i(z_i)$ and $z_0 = T_\beta^j(z_j)$. Hence, $z_0 = T_\beta^j(z_j) = T_\beta^j(z_i) = T_\beta^{j-i}T_\beta^i(z_i) = T_\beta^{j-i}(z_0)$. We thus deduce that $z = z_0$ has a purely periodic β-expansion.

(2) Suppose now that there exist two indices $1 \leq i < j$ such that $z_i = z_j = 1$. We know that $w_{i-1}\dots w_0u_1\dots u_n\cdots = d_\beta^*(1)$ and $w_{j-1}\dots w_0u_1\dots u_n\cdots = d_\beta^*(1)$. This implies that $u_1\dots u_n\dots$ is purely periodic, and that there exists $k \in \mathbb{N}$ such that $(u_i)_{i\geq 1} = S^k(d_\beta^*(1))$. Furthermore, the only case where $d_\beta^*(1)$ is purely periodic corresponds to β simple Parry number.

(3) Suppose that there exists $i \geq 1$ such that $z_i = 1$ and for all $k \neq i$, $z_k < 1$. One has $T_\beta(z_{i+1}) = 0$. Furthermore, the map T_β is onto the stable set $\{z_k;\ k > i\} \cup \{0\}$. Since this set is finite, T_β is also one-to-one. From $T_\beta(z_{i+1}) = 0 = T_\beta(0)$, we deduce that $z_{i+1} = 0$, and similarly, that $z_k = 0$ for all $k > i$. We deduce that $w_k = \beta z_{k+1} - z_k = 0$ for all $k > i$.

    We know that $\delta_\beta(\sum_{i\geq 0} w_i\beta^i) = -\delta_\beta(z)$. In the present case, the power series is indeed a polynomial, therefore $\delta_\beta(w_0 + \cdots + w_i\beta^i) = -\delta_\beta(z)$. We thus have two polynomials in β with coefficients in $\mathbb{Z}$ that coincide on all the conjugates of β. By applying a Galois transformation, we deduce that $z = -(w_0 + \cdots + w_p\beta^p)$. But by construction $z > 0$ and $w_0 + \cdots + w_p\beta^p \geq 0$, which yields a contradiction. ∎

5. Conclusion

This formalism should now be used to further study topological or metrical properties of the sets $\mathcal{R}_\beta$ and $\widetilde{\mathcal{R}_\beta}$, even in the non-Pisot case, which will be the focus of a subsequent paper. Our aims are as follows: first the construction of explicit Markov partitions of endomorphisms of the torus as initiated in [Sie00], secondly, the study of rational numbers having a purely periodic expansion in the same vein as that of [Aki98, Sch80], and thirdly, the spectral study of β-shifts in the Pisot non-unit case according to [Sie03].

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